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Vibration Characteristics of Functionally Graded Fixed Thin Rectangular Plate on Winkler Elastic Foundation Using Beam Analogy Method

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Abstract: **In this study, integral calculus has been applied to the beam analogy method for the evaluation of the non-dimensional frequency parameters of isotropic functionally graded (FG) rectangular plates resting on Winkler elastic foundation. The fundamental assumptions of linear, elastic, small-deflection theory of bending for thin plates due to Kirchhoff are taken into consideration. Using direct integration, characteristic orthogonal polynomials (COPs) shape function for all-round clamped (CCCC) plate is formulated. The effect of aspect ratios on the natural frequency of the plate is examined. It is evident that adding an elastic foundation increases the non-dimensional frequency parameter of the plates. Furthermore, like plates resting on Winkler foundation, an increase in aspect ratio, causes a corresponding increase in frequency for plates not subjected to the effect of Winkler elastic foundation regardless of the boundary configuration. Hence, fixity of supports increases the fundamental natural frequency of the plate, and so increases the resistance of the plate to higher forcing frequencies before resonance can occur. It is also observed that an increase in power law index decreases the frequency parameters of the plate. The validity of the present theory is investigated by comparing some of the present results with those reported in the literature. It can be concluded that the proposed theory is accurate and simple in solving the free vibration behavior of FG plates.**

Keywords: **Winkler foundation, vibration, functionally graded plate, rectangular plat, characteristic orthogonal polynomials.**

I. INTRODUCTION

Composite materials have been extensively used in modern engineering structures, especially in the aerospace industry. These materials have many advantages over the conventional engineering materials, such as low weight, high stiffness-to-weight and strength-to-weight ratios, environmental resistance, and the flexibility in design. A functionally graded material (FGM) is usually a multi-phased material with the volume fractions of its constituents varied gradually along specific directions. Compared with conventional materials, FGMs possess a number of advantages such as reduced residual and thermal stresses, improved bonding strength between dissimilar materials, enhanced environmental resistance and optimized strength.

Elastic foundations are common technical problems in engineering and many solutions have been proposed in recent years. The simplest type of elastic foundation was proposed by Winkler [1]. Winkler's model expressed the relationship between external load and deflection of the foundation surface. The major deficiency of this model is having no interaction between the springs. Pasternak [2] improved the Winkler model by attaching a shear layer to the springs. In the time since, many research works have been conducted to assess the vibration characteristics of plates on elastic foundation. In the time since, many research works have been conducted to assess the vibration characteristics of plates on elastic foundation.

A new version of the differential quadrature method for assessing the vibration characteristics of rectangular plates resting on elastic foundations carrying any number of sprung masses was proposed by Hsu [3]. The first six natural frequencies of plates with various foundation stiffnesses were highlighted. They also analyzed the effect of aspect ratios on the natural frequency of plates on elastic foundation. Using the finite cosine integral transform method, Li et al. [4] presented the analytical solutions for rectangular plates on the Winkler elastic foundation with four edges free. In the analysis, the classical Kirchhoff rectangular plate was considered. Chakraverty and Pradhan [5] investigated the free vibration of functionally graded (FG) rectangular plates subject to different sets of boundary conditions within the framework of classical plate theory. The parametric resonance characteristics of functionally-graded material (FGM) plates on elastic foundation under biaxial in plane periodic load was studied by Ramu and Mohanty [6]. Finite element method and Hamilton's principle were utilized to establish the governing equations in a discrete form. Floquet's theory was applied to determine the instability regions of FGM plate resting on elastic foundation. The three-dimensional vibration of a functionally graded sandwich rectangular plate on an elastic foundation with normal boundary conditions was analyzed by Cui et al. [7] using a semi-analytical method based on three-dimensional elasticity theory. In their study, Kumar et al. [8] investigated the free vibration behaviour of thin functionally graded rectangular plates by using the dynamic stiffness method (DSM). Zhao-chun et al. [9] more recently assessed the free vibration characteristics of porous functionally graded material (FGM) rectangular plates on a Winkler-Pasternak elastic foundation under the influence of temperature based on the classical thin plate theory and Hamilton principle. The Voigt mixed power law model and random distribution model of pores were used to characterize the material properties of porous FGM rectangular plates, and the uniform temperature rise in a porous FGM rectangular plate and the temperature dependency of material properties were considered.

It's crucial to quickly and precisely examine the plate vibration properties. Inadequate calculations and consideration of deflections and natural frequencies can result in safety and cost-saving mistakes as well as the complete collapse of a structure, particularly when the vibration reaches resonance. Therefore, it is necessary to create a quicker and easier method that precisely predicts how functionally graded plates with different boundary conditions would behave on an elastic basis. This is the primary focus of the current study.

II. THEORETICAL BACKGROUND

The formulation of the exact solution to the governing differential equation of the plate studied, development of the characteristic orthogonal shape functions and the fundamental natural frequencies of all round clamped plate (CCCC) with various aspect ratios are presented.

Formulation of an Exact Solution to the Governing Differential Equation

According to the theory of the classical Kirchhoff plate, the governing equation of motion for an unloaded plate on an elastic foundation is:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{k}{D} w(x,y,t) + \frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

where

$$D = \frac{Eh^3}{12(1-\upsilon^2)} \tag{2}$$

D = flexural rigidity of plate

E = Young's moduli

ν = Poisson's ratio

h = thickness of plate

ρ = density of plate

w(x, y, t) = out-of-plane displacement

k = reaction coefficient of foundation

Assuming a harmonic vibration, we may write:

$$\mathrm{w(x,y,t)} + W(x,y)sin\omega t \tag{3}$$

Where $(x,\ y)$ is the shape function describing the modes of vibration and w is the natural circular frequency of the plate.

$$\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) w(x,y,t) + \frac{k}{D} w(x,y,t) + \frac{\rho h}{D}\left(\frac{\partial^2 w}{\partial t^2}\right)(x,y,t) = 0 \tag{4}$$

Substituting Equation (3) into Equation (4) gives

$$sin\omega t\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) w(x,y) + \frac{k}{D} w(x,y) sin\omega t\ - \frac{\rho h}{D}\omega^2 w sin\omega t = 0 \tag{5}$$

Dividing Equation (5) by $sin\omega t$ we have

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{kw}{D} - \frac{\rho h \omega^2 w}{D} = 0 \tag{6}$$

where w represents $w(x,\ y)$

Multiplying Equation (6) by w, we have

$$\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) w + \frac{kw^2}{D} - \frac{\rho h \omega^2 w^2}{D} = 0 \tag{7}$$

Integrating both sides of Equation (7)

$$\int_0^a \int_0^b \left(\left(\frac{\partial^4 w}{\partial x^4}\right) w + \left(2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) w + \left(\frac{\partial^4 w}{\partial y^4}\right) w + \frac{kw^2}{D} - \frac{\rho h \omega^2 w^2}{D}\right) \partial x \partial y = 0 \tag{8}$$

But

$$\zeta = \frac{x}{a}$$

$$\eta = \frac{y}{b}$$

After substituting ζ and η, Equation (8) becomes

$$\int_0^1 \int_0^1 \left(\left(\frac{\partial^4 w}{a^4 \partial \zeta^4}\right) w + \frac{2b^2}{a^2}\left(\frac{\partial^4 w}{\partial \zeta^2 \partial \eta^2}\right) w + \frac{1}{b^4}\left(\frac{\partial^4 w}{\partial \eta^4}\right) w + \frac{kw^2}{D} - \frac{\rho h \omega^2 w^2}{D}\right) ab\, \partial \zeta \partial \eta = 0 \tag{9}$$

where $\partial x = a\partial\zeta$ and $\partial y = b\partial\eta$

Multiplying Equation (9) by b^4, we have

$$\frac{ab}{b^4}\int_0^1 \int_0^1 \left(\left(\frac{b^4 \partial^4 w}{a^4 \partial \zeta^4}\right) w + \frac{2b^2}{a^2}\left(\frac{\partial^4 w}{\partial \zeta^2 \partial \eta^2}\right) w + \left(\frac{\partial^4 w}{\partial \eta^4}\right) w + \frac{b^4 kw^2}{D} - \frac{b^4 \rho h \omega^2 w^2}{D}\right) \partial \zeta \partial \eta = 0 \tag{10}$$

Substituting $\beta = \frac{a}{b}$ into Equation (10), we have

$$\int_0^1 \int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial \zeta^4}\right) w + \frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial \zeta^2 \partial \eta^2}\right) w + \left(\frac{\partial^4 w}{\partial \eta^4}\right) w + \frac{b^4 kw^2}{D} - \frac{b^4 \rho h \omega^2 w^2}{D}\right) \partial \zeta \partial \eta = 0 \tag{11}$$

$$\int_0^1\int_0^1\left(\left(\frac{1}{\beta^4}\left(\frac{\partial^4 w}{\partial\zeta^4}\right)w+\frac{2}{\beta^2}\left(\frac{\partial^4 w}{\partial\zeta^2\partial\eta^2}\right)w+\left(\frac{\partial^4 w}{\partial\eta^4}\right)w\right)\partial\zeta\partial\eta\ +\int_0^1\int_0^1\frac{b^4kw^2}{D}\partial\zeta\partial\eta\ -\int_0^1\int_0^1\frac{b^4\rho h\omega^2w^2}{D}\right)\partial\zeta\partial\eta = 0 \quad (12)$$

But

$$w(x,y)=w(\zeta,\eta)=AS_p \quad (13)$$

Substituting Equation (13) into (12), we have

$$\int_0^1\int_0^1\left(\left(\frac{1}{\beta^4}\left(\frac{\partial^4(AS_p)}{\partial\zeta^4}\right)AS_p+\frac{2}{\beta^2}\left(\frac{\partial^4(AS_p)}{\partial\zeta^2\partial\eta^2}\right)AS_p+\left(\frac{\partial^4(AS_p)}{\partial\eta^4}\right)AS_p\right)\partial\zeta\partial\eta\ +\int_0^1\int_0^1\frac{b^4k(AS_p)^2}{D}\partial\zeta\partial\eta - \int_0^1\int_0^1\frac{b^4\rho h\omega^2(AS_p)^2}{D}\right)\partial\zeta\partial\eta=0 \quad (14)$$

Since A is a constant, Equation (14) can be re-written as

$$A^2\int_0^1\int_0^1\left(\left(\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\zeta^4}\right)S_p+\frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\partial\eta^2}\right)S_p+\left(\frac{\partial^4 S_p}{\partial\eta^4}\right)S_p\right)\partial\zeta\partial\eta\ +A^2\int_0^1\int_0^1\frac{b^4k{S_p}^2}{D}\partial\zeta\partial\eta - \int_0^1\int_0^1\frac{b^4\rho h\omega^2{S_p}^2}{D}\right)\partial\zeta\partial\eta\ =0 \quad (15)$$

$$Let\ J_2=\left[\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\zeta^4}\right)S_p+\frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\partial\eta^2}\right)S_p+\left(\frac{\partial^4 S_p}{\partial\eta^4}\right)\right] \quad (16)$$

Substituting Equation (16) into Equation (15) yields

$$A^2\int_0^1\int_0^1(J_2S_p)\partial\zeta\partial\eta+\ A^2\int_0^1\int_0^1\frac{b^4k{S_p}^2}{D}\partial\zeta\partial\eta-A^2\int_0^1\int_0^1\frac{b^4\rho h\omega^2{S_p}^2}{D}\partial\zeta\partial\eta=0 \quad (17)$$

$$Let\ B_2=\int_0^1\int_0^1{S_p}^2\partial\zeta\partial\eta \quad (18)$$

and

$$Let\ C_2=\int_0^1\int_0^1(J_2S_p)\partial\zeta\partial\eta \quad (19)$$

$$A^2C_2+A^2B_2\frac{b^4k}{D}-A^2B_2\frac{b^4\rho h\omega^2}{D}=\ 0 \quad (20)$$

Dividing Equation (20) by A^2, we have

$$C_2+B_2\frac{b^4k}{D}-B_2\frac{b^4\rho h\omega^2}{D}=\ 0 \quad (21)$$

Dividing Equation (21) by B_2 yields

$$\frac{C_2}{B_2}+\frac{b^4k}{D}\ -\ \frac{b^4\rho h\omega^2}{D}=\ 0 \quad (22)$$

$$\frac{b^4\rho h\omega^2}{D} = \frac{C_2}{B_2} + \frac{b^4 k}{D} \tag{23}$$

$$\rho h\omega^2 = \left(\frac{C_2}{B_2} + \frac{b^4 k}{D}\right)\frac{D}{b^4} \tag{24}$$

$$\rho h\omega^2 = \frac{C_2}{B_2}\frac{D}{b^4} + \mathrm{k} \tag{25}$$

$$\omega^2 = \frac{1}{b^4}\frac{C_2}{B_2}\frac{D}{\rho h} + \mathrm{k} \tag{26}$$

$$\omega = \frac{1}{b^2}\sqrt{\frac{C_2}{B_2}}\sqrt{\frac{D}{\rho h}} + \sqrt{k} \tag{27}$$

$$\omega = \frac{\sqrt{\frac{C_2}{B_2}}}{b^2}\sqrt{\frac{D}{\rho h}} + \sqrt{k} \tag{28}$$

$$\omega = \frac{\sqrt{\frac{A_1}{\beta^4} + \frac{B_1}{\beta^2} + C_1}}{b^2}\sqrt{\frac{D}{\rho h}} + \sqrt{k} \tag{29}$$

where

$\beta = \frac{a}{b}\,, \qquad \frac{C_2}{B_2} = \frac{A_1}{\beta^4} + \frac{B_1}{\beta^2} + C_1$

A_1, B_1, C_1 are numerical coefficients that arise after evaluating the ratio $\frac{C_2}{B_2}$

The use of dimensionless coordinates, ζ and η, has simplified the evaluation of the integrals by making the limits of integration to run from 0 to 1.

Now for each value of β, the equation of ω is obtained in the form

$$\omega = \frac{H_{b\beta}}{b^2}\sqrt{\frac{D}{\rho h}} + \sqrt{k} \tag{30}$$

Rearranging Equation (30) and making $H_{b\beta}$ the subject of the equation gives

$$H_{b\beta} = \omega b^2\sqrt{\frac{\rho h}{D}} - \sqrt{k} \tag{31}$$

Where $H_{b\beta}$ is a numerical coefficient called the non-dimensional frequency parameter, expressed in terms of the dimension 'b' for a plate of aspect ratio $\beta = \frac{a}{b}$

It is clear from Equations (29) and (30) that

$$H_{b\beta} = \sqrt{\frac{A_1}{\beta^4} + \frac{B_1}{\beta^2} + C_1} \tag{32}$$

In order to obtain the non-dimensional frequency parameter, $H_{b\beta}$, for a plate of inverse aspect ratio, $\varphi = \frac{b}{a}$, $\frac{1}{\varphi}$ is substituted for β in Equations (29) and (32) to get Equations (33) and (34) respectively

$$\omega = \frac{\sqrt{A_1\varphi^4 + B_1\varphi^2 + C_1}}{b^2}\sqrt{\frac{D}{\rho h} + \sqrt{k}} \qquad (33)$$

$$H_{b\beta} = \sqrt{A_1\varphi^4 + B_1\varphi^2 + C_1} \qquad (34)$$

Beam Analogy Method

The beam analogy method is used in this research work to carry out dynamic analysis of rectangular, thin, isotropic plates because of the following advantages it offers. It is simple and straight-forward; it is not rigorous or complex and does not involve complicated mathematics. It uses basic principles like partial differentiation and integration, work, conservation of energy, beam flexure, plate geometry and material properties of the plate. Evaluation of integrals is further simplified by the use of a dimensionless co-ordinate system, ζ and η which makes the limits of integrals to run from zero to unity.

The beam analogy method makes use of characteristic orthogonal polynomials (COPs), (Chakraverty [10]) to obtain meaningful deflection shape functions for each beam strip and plate analyzed in this work. The detailed steps for development of characteristic orthogonal polynomials are highlighted in this study. Chakraverty [10] gave assurance that when COPs are used the numerical solutions converge to the exact solution.

The differential equation of motion of the plate (Equation (1)) is a fourth order equation in x and y co-ordinates. When expressed in dimensionless form, it boils down to a fourth order equation in ζ and η co-ordinates. So, its solution, which is the deflection shape function, will be a fourth degree polynomial in ζ and η. Since the governing differential equation of motion given in this study is a fourth order equation, then the shape function for each beam strip is likely to be a fourth degree polynomial. So, assuming the beam strips in ζ and η directions have the following deflection shape functions:

$$w_{(\zeta)} = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 \qquad (35)$$

$$w_{(\eta)} = b_0 + b_1\eta + b_2 + b_3\eta^3 + b_4\eta^4 \qquad (36)$$

The free edge of a plate has three boundary conditions (i.e., shear, bending moment and twisting moment are zeros), instead of two boundary conditions. To take care of this extra boundary condition, the shape function must contain an additional term for the problem to be statically determinate. So, for a beam strip having one free end, a fifth degree polynomial is used, thus:

$$w_{(\zeta)} = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^5 \qquad (37)$$

$$w_{(\eta)} = b_0 + b_1\eta + b_2 + b_3\eta^3 + b_4\eta^4 + b_5\eta^5 \qquad (38)$$

The values of a_1 to a_4 and b_1 to b_4 are determined with the aid of the beam's boundary conditions, as presented in this study. Naturally, since the plate is a two-dimensional array of both beam strips (in ζ and η directions), the deflection shape functions for the plates were obtained from the following expression:

$$w_{(\zeta,\eta)} = w_{(\zeta)}.w_{(\eta)} \qquad (39)$$

Development of Characteristic Orthogonal Polynomials (COPs)

Let us consider a rectangular plate of dimensions, a along x and b along y, of uniform thickness shown in Figure 1. If the deflection pattern of the plate along x is represented by a beam strip qualitatively, the beam function along x is taken as F(x). Similarly, the corresponding beam function along y is taken as F(y).

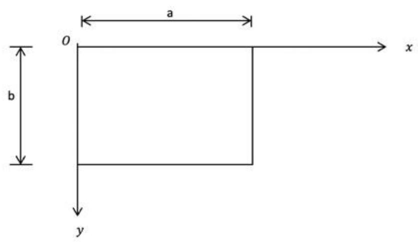


Figure 1: A rectangular plate

Assuming the plate deflections in the form of a series, the solution for prismatic beam of constant stiffness EI and length spanning along x can be written as:

$$w_x = F(x) = \sum_{m=1}^{\infty} X_m x^m \tag{40}$$

and in the y-direction,

$$w_y = F(y) = \sum_{n=1}^{\infty} Y_n y^n \tag{41}$$

Where,

w_x and w_y are plate deflections at point (x,y)

X_m and Y_n are constant parameters in x and y directions respectively

x, y are coordinates of points

m and n are series to infinity limit

F(x) and F(y) are beam functions along x and y directions respectively

Bhat [11] developed a systematic method of constructing the shape function of rectangular plates using the characteristic orthogonal polynomial by assuming the displacement function as a product of two functions: one which is a pure function of x and the other is of y so that,

$w(x,y) = F(x).F(y) = w_x.w_y$

or

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_m x^m Y_n y^n \tag{42}$$

Expressing Equations (40), (41), (42) in the form of non-dimensional parameters, R and Q , Equation (40) becomes

$$w_x = F(x) = \sum_{m=1}^{\infty} X_m \ (aR)^m \quad = \sum_{m=1}^{\infty} X_m a^m R^m \tag{43}$$

In the same manner, substituting y = bQ into Equation (41), we have:

$$w_y = F(y) = \sum_{n=1}^{\infty} Y_n \ (bQ)^n \quad = \sum_{n=1}^{\infty} Y_n b^n Q^n \tag{44}$$

Substituting Equations (42) and (43) into Equation (42) we obtain:

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_m a^m R^m \ Y_n b^n Q^n \tag{45}$$

or

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_m R^m B_n \ Q^n \tag{46}$$

where

A_m and B_n are coefficients that are to be determined from the boundary conditions at the edges of the plate.

The equation of an orthotropic plate in free vibration is a fourth order differential, the density of the plate being constant. Therefore, m and n in Equation (46) must be equal to 4, Onyeyili [12]. Expanding Equations (44), (45) and (46) to 4th order power series, we obtain

$$w_x = F(x) = \sum_{m=1}^{4} A_m R^m = A_0 + A_1R + A_2R^2 + A_3R^3 + A_4R^4 \quad (47)$$

$$w_y = F(y) = \sum_{n=1}^{4} B_n Q^n = B_0 + B_1Q + B_2Q^2 + B_3Q^3 + B_4Q^4 \quad (48)$$

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_m R^m B_n Q^n = F(x).G(y)$$

$$w(x, y) = (A_0 + A_1R + A_2R^2 + A_3R^3 + A_4R^4)(B_0 + B_1Q + B_2Q^2 + B_3Q^3 + B_4Q^4) \quad (49)$$

The bending moments of plate in x and y directions are given as:

$$M_x = \frac{-D_x\,\partial^2 w}{\partial x^2} \quad (50)$$

$$M_y = \frac{-D_y\,\partial^2 w}{\partial y^2} \quad (51)$$

where D_x and D_y are flexural rigidities of the plate in the x and y directions.

Substituting into Equations (50) and (51) w_x and w_y from Equations (47) and (48), Mx and My can be non-dimensionalized into the following expression

$$M_x = (2A_2 + 6A_3R + 12A_4R^2)\frac{-Dx}{a^2} \quad (52)$$

and

$$M_y = (2B_2 + 6B_3Q + 12B_4Q^2)\frac{-Dy}{b^2} \quad (53)$$

Equations (47), (48), (52), and (53) are used to obtain the displacement functions of the plate.

Boundary Conditions

If a plate is clamped at the boundary, then the deflection and the slope of the middle surface must vanish at the boundary. On a clamped edge parallel to the y axis at x = a, the boundary conditions are

$$w|_{x=a} = 0 \quad (54)$$

$$\frac{\partial w}{\partial x}|_{x=a} = 0 \quad (55)$$

The boundary conditions on the clamped edge parallel to the x axis at y = b are

$$w|_{y=b} = 0 \quad (56)$$

$$\frac{\partial w}{\partial x}|_{y=b} = 0 \quad (57)$$

Development of Shape Function for CCCC Plate

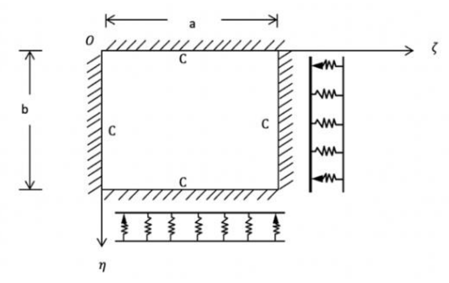


Figure 2: Thin rectangular plate whose edges are clamped (CCCC) resting on Winkler foundation

Boundary conditions

- Deflections at all edges are zero
- Slope at all edges is zero

For $\zeta -$ directions

At $\zeta = 0$

From Equation (47)

$W_x = A_0 = 0$

$\therefore A_o = 0$ (58)

$\frac{\partial W_\zeta}{a\partial\zeta} = \frac{\partial W_x}{a\partial x}(A_0 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + A_4\zeta^4)$

$\frac{\partial W_\zeta}{a\partial\zeta} = \frac{A_1+2A_2\zeta+3A_3\zeta^2+4A_4\zeta^3}{a}$ (59)

Then,

At $\zeta = 0$

$\frac{\partial W_\zeta}{\partial\zeta} = 0 = A_1 = 0$

$$\therefore A_1 = 0 \quad (60)$$

At $\zeta = 1$

$W_X = 0 = A_2 + A_3 + A_4$

$$\therefore A_2 = -(A_3 + A_4) \quad (61)$$

From Equation (59),

$\frac{\partial W_\zeta}{\partial\zeta} = 2A_2 + 3A_3 + 4A_4 = 0$

Substituting $A_2 = -(A_3 + A_4)$, we obtain

$-2(A_3 + A_4) + 3A_3 + 4A_4 = 0$

$A_3 + 2A_4 = 0$

$$\therefore A_3 = -2A_4 \quad (62)$$

$$Thus, \quad A_2 = -(-2A_4 + A_4) = A_4 \quad (63)$$

Putting the values of $A_0, A_1, A_{2,}A_3$, into (47) we have

$$W_x = \mathrm{F}(\zeta) = A_4\zeta^2 + (-2A_4)\zeta^3 + A_4\zeta^4 = A_4(\zeta^2 - 2\zeta^3 + \zeta^4) \quad (64)$$

For $\eta - diretion$

At $\eta = 0$

From Equation (48),

$W_y = B_0 = 0$

$$\therefore B_0 = 0 \quad (65)$$

$$\frac{\partial W_y}{\partial y} = \frac{\partial W_y}{b\partial\eta} = \frac{1}{b}(B_1 + 2B_2\eta + 3B_3\eta^2 + 4B_4\eta^3) \quad (66)$$

$$\frac{\partial W_y}{\partial y} = 0 = \frac{1}{b} B_1 = 0$$

$$since\ \frac{1}{b} \neq 0$$

$$B_1 = 0 \tag{67}$$

At $Q = 1$

$$W_y = G(\eta = 1) = 0 = B_2 + B_3 + B_4 = 0$$

$$\Longrightarrow B_2 + B_3 + B_4 = 0 \tag{68}$$

$$B_2 = -(B_3 + B_4) \tag{69}$$

From Equation (66)

$$\frac{\partial W_y}{\partial_y} = 0 = \frac{1}{b}(2B_2 + 3B_3 + 4B_4) = 0$$

$$since\ \frac{1}{b} \neq 0$$

$$\Longrightarrow \quad (2B_2 + 3B_3 + 4B_4) = 0 \tag{70}$$

Putting the value of B_2 of Equation (69) into Equation (70), we obtain

$$-2(B_3 + B_4) + 3B_3 + 4B_4 = 0$$

$$\Longrightarrow B_3 + 2B_4 = 0$$

$$\therefore\ B_3 = -2B_4 \tag{71}$$

Putting the expression of (71) into (69) we obtain

$$B_2 = -(B_3 + B_4)$$

$$\Longrightarrow \quad B_2 = -(-2B_4 + B_4)$$

$$\therefore \quad B_2 = -(-B_4)$$

$$B_2 = B_4 \tag{72}$$

Putting the expression of $B_0, B_1, B_2 and\ B_4$ into Equation (48), we have

$$w_y = G(\eta) = B_4\eta^2 + (-2B_4)\eta^3 + B_4\eta^4$$

$$= \ B_4(\eta^2 - 2\eta^3 + \eta^4) \tag{73}$$

Multiplying Equations (64) and (73) we obtain the displacement function for a rectangular plate clamped supported all round as;

$$w(\zeta,\eta) = \mathrm{F}(\zeta) * G(\eta) = W_x * W_y$$

$$w(\zeta,\eta) = A_4(\ \zeta^2 - 2\ \zeta^3 +\ \zeta^4) * B_4(\eta^2 - 2\eta^3 + \eta^4)$$

$$w(\zeta,\eta) = A_4B_4(\ \zeta^2 - 2\ \zeta^3 +\ \zeta^4)(\eta^2 - 2\eta^3 + \eta^4)$$

$$w(\zeta,\eta) = K(\ \zeta^2 - 2\ \zeta^3 +\ \zeta^4)(\eta^2 - 2\eta^3 + \eta^4) \tag{74}$$

Development of Fundamental Natural Frequency Expression of CCCC Plate for Free Vibration

w(x, y) = w (ζ , η) = kSp

where: k = deflection constant

Sp = a polynomial in ζ and η

From Equation (74) we have

$$w(\zeta,\eta) = S_p = (\zeta^2 - 2\zeta^3 + \zeta^4)(\eta^2 - 2\eta^3 + \eta^4) \quad (75)$$

$$\frac{\partial^4 S_p}{\partial \zeta^4} = 24(\eta^2 - 2\eta^3 + \eta^4)$$

$$\frac{\partial^4 S_p}{\partial \eta^4} = 24(\zeta^2 - 2\zeta^3 + \zeta^4)$$

$$\frac{\partial^4 S_p}{\partial \zeta^2 \partial \eta^2} = 4(1 - 6\zeta - 6\zeta^2)(1 - 6\eta + 6\eta^2)$$

$$K_2 = \frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial \zeta^4}\right) + \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial \zeta^2 \partial \eta^2}\right) + \left(\frac{\partial^4 S_p}{\partial \eta^4}\right)$$

$$= \frac{1}{\beta^4}24(\eta - 2\eta^3 + \eta^4) + \frac{2}{\beta^2}4(1 - 6\zeta - 6\zeta^2)(1 - 6\eta + 6\eta^2) + 24(\zeta^2 - 2\zeta^3 + \zeta^4)$$

But

$$\left(\frac{\partial^4 S_p}{\partial \zeta^4}\right) S_p = 24(\eta^2 - 2\eta^3 + \eta^4)(\zeta^2 - 2\zeta^3 + \zeta^4)(\eta^2 - 2\eta^3 + \eta^4)$$

$$= 24(\zeta^2 - 2\zeta^3 + \zeta^4)(\eta^4 - 4\eta^5 + 6\eta^6 - 4\eta^7 + \eta^8)$$

$$\left(\frac{\partial^4 S_p}{\partial \eta^4}\right) S_p = 24(\zeta^2 - 2\zeta^3 + \zeta^4)(\zeta^2 - 2\zeta^3 + \zeta^4)(\eta^2 - 2\eta^3 + \eta^4)$$

$$= 24(\eta^2 - 2\eta^3 + \eta^4)(\zeta^4 - 4\zeta^5 + 6\zeta^6 - 4\zeta^7 + \zeta^8)$$

$$\left(\frac{\partial^4 S_p}{\partial \zeta^2 \partial \eta^2}\right) S_p = 4(1 - 6\zeta - 6\zeta^2)(1 - 6\eta + 6\eta^2)(\zeta - 2\zeta^3 + \zeta^4)(\eta - 2\eta^3 + \eta^4)$$

$$= 4(\zeta^2 - 8\zeta^3 + 19\zeta^4 - 18\zeta^5 + 6\zeta^5)(\eta^2 - 8\eta^3 + 19\eta^4 - 18\eta^5 + 6\eta^6)$$

Recall that

$$S_p = (\zeta^2 - 2\zeta^3 + \zeta^4)(\eta^2 - 2\eta^3 + \eta^4)$$

Therefore

$${S_p}^2 = (\zeta^4 - 4\zeta^5 + 6\zeta^6 - 4\zeta^7 + \zeta^8)(\eta^4 - 4\eta^5 + 6\eta^6 - 4\eta^7 + \eta^8) \quad (76)$$

Now

$$\int_0^1 \int_0^1 \left(\frac{\partial^4 S_p}{\partial \zeta^4}\right) S_p \, \partial\zeta \partial\eta =$$

$$\int_0^1 \int_0^1 24(\zeta - 2\zeta^3 + \zeta^4)(\eta^2 - 4\eta^4 + 2\eta^5 + 4\eta^6 - 4\eta^7 + \eta^8) \, \partial\zeta \partial\eta$$

$$= 24\left(\frac{1}{2} - \frac{2}{4} + \frac{1}{5}\right)\left(\frac{1}{5} - \frac{2}{3} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9}\right)$$

$$= 0.001269845$$

Therefore,

$$\int_0^1 \int_0^1 \left(\frac{\partial^4 S_p}{\partial \zeta^4}\right) S_p \, \partial\zeta \partial\eta = 0.001269845 \frac{1}{\beta^4}$$

$$\int_0^1\int_0^1 \left(\frac{\partial^4 S_p}{\partial\zeta^2\partial\eta^2}\right) S_p\,\partial\zeta\partial\eta$$

$$= 4(\zeta^2 - 8\zeta^3 + 19\zeta^4 - 18\zeta^5 + 6\zeta^5)\,(\eta^2 - 8\eta^3 + 19\eta^4 - 18\eta^5 + 6\eta^6)\partial\zeta\partial\eta$$

$$= 4\left(\frac{1}{3} - 2 + \frac{19}{5} - 3 + \frac{6}{7}\right)\left(\frac{1}{3} - 2 + \frac{19}{5} - 3 + \frac{6}{7}\right)$$

$$= 0.00036281175$$

$$\int_0^1\int_0^1 \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\partial\eta^2}\right) S_p\,\partial\zeta\partial\eta = 0.0007257235\frac{1}{\beta^2}$$

$$\int_0^1\int_0^1 \left(\frac{\partial^4 S_p}{\partial\eta^4}\right) S_p\,\partial\zeta\partial\eta = 0.001269845$$

$$C_2 = \int_0^1\int_0^1 (K_2 S_p)\,\partial\zeta\partial\eta$$

$$C_2 = \int_0^1\int_0^1 \left(\frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\eta^4}\right) + \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\partial\eta^2}\right) + \left(\frac{\partial^4 S_p}{\partial\eta^4}\right)\right) S_p\partial\zeta\partial\eta$$

$$= \int_0^1\int_0^1 \frac{1}{\beta^4}\left(\frac{\partial^4 S_p}{\partial\eta^4}\right) S_p\partial\zeta\partial\eta + \int_0^1\int_0^1 \frac{2}{\beta^2}\left(\frac{\partial^4 S_p}{\partial\zeta^2\partial\eta^2}\right) S_p\partial\zeta\partial\eta + \int_0^1\int_0^1 \left(\frac{\partial^4 S_p}{\partial\eta^4}\right) S_p\partial\zeta\partial\eta$$

$$C_2 = 0.001269845\frac{1}{\beta^4} + 0.0007257235\frac{1}{\beta^2} + 0.001269845$$

$$\int_0^1\int_0^1 {S_p}^2\,\partial\zeta\partial\eta = \int_0^1\int_0^1 (\zeta^4 - 4\zeta^5 + 6\zeta^6 - 4\zeta^7 + \zeta^8)(\eta^4 - 4\eta^5 + 6\eta^6 - 4\eta^7 + \eta^8)\partial\zeta\partial\eta$$

$$= \left(\frac{1}{5} - \frac{4}{6} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9}\right)\left(\frac{1}{5} - \frac{4}{6} + \frac{6}{7} - \frac{4}{8} + \frac{1}{9}\right)$$

$$= 0.00000251953$$

$$B_2 = \int_0^1\int_0^1 {S_p}^2\,\partial\zeta\partial\eta = 0.00000251953$$

$$\frac{C_2}{B_2} = \frac{0.001269845\frac{1}{\beta^4} + 0.0007257235\frac{1}{\beta^2} + 0.001269845}{0.00000251953}$$

$$= \frac{504.1067884}{\beta^4} + \frac{288.0603414}{\beta^2} + 504.1067884$$

From Equation (28)

$$\omega = \frac{\sqrt{\frac{504.1067884}{\beta^4} + \frac{288.0603414}{\beta^2} + 504.1067884}}{b^2}\sqrt{\frac{D}{\rho h}} + \sqrt{k} \qquad (77)$$

Comparing Equations (33) and (77) we have

$$H_{b\beta} = \sqrt{\frac{504.1067884}{\beta^4} + \frac{288.0603414}{\beta^2} + 504.1067884}$$

From Equations (34) we have

$$H_{b\beta} = \sqrt{504.1067884\varphi^4 + 288.0603414\varphi^2 + 504.1067884} \quad (78)$$

Functionally Graded Plate

A functionally graded plate with length *a*, width *b* and a uniform thickness *h* is considered. The geometry of the plate and the coordinate system are shown in Figure 3.

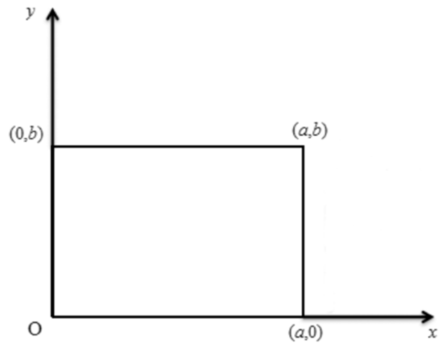


Figure 3: A typical FG rectangular plate element in Cartesian coordinates

It is assumed that the material properties of the FG plate vary smoothly through the thickness. Based on the volume fraction of the constituent material, the Young's modulus and density of FG plate can be written as functions of thickness coordinate, z, as follows (Birman and Byrd [15]):

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^n + E_m \quad (79)$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^n + \rho_m \quad (80)$$

where *n* is the power law index of the FG rectangular plate, the subscripts m and c show the metal and ceramic surfaces, respectively. Due to the small variations of the Poisson's ratio, ν, it is assumed to be constant Chakraverty and Pradhan [5].

According to this distribution, the bottom surface ($z = -\frac{h}{2}$) of FG plate pure metal, whereas the top surface ($z = \frac{h}{2}$) is pure ceramic. The stiffness coefficient is

$$D = D_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\, Z^2 dz \quad (81)$$

$$= \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{E(z)}{1 - \nu^2} Z^2 dz \quad (82)$$

$$= \frac{1}{1 - \nu^2}\int_{\frac{-h}{2}}^{\frac{h}{2}} E(z)\, Z^2 dz \quad (83)$$

$$= \frac{1}{1 - \nu^2}\int_{\frac{-h}{2}}^{\frac{h}{2}} \left\{(E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^n + E_m\right\} Z^2 dz \quad (84)$$

$$= \frac{1}{1-\nu^2}\int_{\frac{-h}{2}}^{\frac{h}{2}} \left\{(E_c - E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^n\right\} Z^2 dz + \int_{\frac{-h}{2}}^{\frac{h}{2}} E_m Z^2 dz \quad (85)$$

$$= \frac{(E_c - E_m)h^3}{1-\nu^2}\left\{\frac{1}{n+3} - \frac{1}{n+2} + \frac{1}{4(n+1)}\right\} + \frac{E_m h^3}{12(1-\nu^2)} \quad (86)$$

While the inertia coefficient is

$$I_0 = \rho h = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho(z)\ dz \quad (87)$$

$$= \frac{(\rho_c - \rho_m)h}{n+1} + \rho_m h \quad (88)$$

Material Properties of the FGM constituents

An Al/Al_2O_3 functionally graded plate which is composed of aluminum (as metal) and alumina (as ceramic) is considered. The Young's modulus and density of aluminum are E_m = 70 GPa and ρ_m = 2700 kg/m^3, respectively, and that of alumina are E_c = 380 GPa and ρ_c = 3800 kg/m^3, respectively. The Poisson ratio of the plate is assumed to be constant through the thickness and equal to 0.3.

Table 1: Material Properties of the FGM constituents

Properties	Unit	Aluminum (Al)	Alumina (Al_2O_3)
E	GPa	70	380
ρ	Kg/m^3	2700	3800
ν	-	0.3	0.3

III. RESULTS AND DISCUSSION

The results obtained from the preceding section are highlighted here. An Al/Al_2O_3 functionally graded plate which is composed of aluminum (as metal) and alumina (as ceramic) is considered. The Young's modulus and density of aluminum are E_m = 70 GPa and ρ_m = 2700 kg/m^3, respectively, and that of alumina are E_c = 380 GPa and ρ_c = 3800 kg/m^3, respectively. The Poisson ratio of the plate is assumed to be constant through the thickness and equal to 0.3.

The expression for the fundamental natural frequencies of the plate is given as

$$\omega = \frac{\sqrt{A_1\varphi^4 + B_1\varphi^2 + C_1}}{b^2}\sqrt{\frac{D}{\rho h}} + \sqrt{k}$$

The equivalent Winkler parameter is defined as

$$k = \frac{K_w b^4}{D}$$

While the natural frequency equation for the CCCC plate in terms of φ and b is

$$H_{b\beta} = \sqrt{504.1067884\varphi^4 + 288.0603414\varphi^2 + 504.1067884}$$

Table 2 shows the non-dimensional natural frequencies $H_{b\beta}$ for isotropic CCCC plate with various aspect ratios $\beta = \frac{b}{a}$

Table 2: Non-dimensional natural frequencies $H_{b\beta}$ for isotropic CCCC plate with various aspect ratios $\beta = \frac{b}{a}$

k_w	k_s	Aspect Ratio $\beta = \frac{b}{a}$	$H_{b\beta}$
0	0	0.1	22.517
		0.2	22.725

		0.3	23.111
		0.4	23.730
		0.5	24.650
		0.6	25.945
		0.7	27.682
		0.8	29.916
		0.9	32.683
		1.0	36.004
100	0	0.1	25.391
		0.2	25.599
		0.3	25.985
		0.4	26.604
		0.5	27.524
		0.6	28.819
		0.7	30.556
		0.8	32.790
		0.9	35.557
		1.0	37.350

In Table 2, frequency parameters for clamped (CCCC) plates are validated with different aspect ratios viz. 0.2, 0.4, 0.5, 0.7, 1.0, 1.5 and 2.0.

Table 3: Comparison of non-dimensional frequency parameters $H_{b\beta}$ for clamped (CCCC) plates for various aspect ratios

		Non-Dimensional Frequency Parameter			
(K_W, K_S)	Aspect Ratio $\beta = \frac{b}{a}$	Present study	Chakraverty and Pradhan [5]	Yang and Shen [13]	Liu and Liew [14]
(0, 0)	0.2	22.725	22.633	-	-
	0.4	23.730	23.647	-	-
	0.5	24.650	24.579	-	-
	0.7	27.682	27.008	-	-
	1.0	36.003	36.000	35.988	35.938
	1.5	60.863	60.768	-	-
	2.0	98.600	98.318	-	-
(100, 0)	1.0	38.877	37.350	-	-

The present results for frequency parameters are in excellent agreement with open literature. Results obtained are in tandem with the those obtained by Chakraverty and Pradhan [5] for aspect ratios 0.2 to 1.0 and those of Yang and Shen [13] as well as Liu and Liew [14]for aspect ratio 1. It can be deduced from Table 2 that the Winkler foundation parameter has a dominant influence on the frequencies of plates on elastic foundation. Without considering the effect of Winkler elastic foundation, an increase in aspect ratios leads to increase in frequency parameters.

Table 4: The frequency parameters of CCCC FG rectangular plates with different *n* and aspect ratios, (K_w = 0)

Power-law exponent *n*	Aspect ratio (*b/a*)	Present study	Chakraverty and Pradhan [5]
0	0.2	22.725	22.633
	0.5	24.650	24.579
	1.0	36.003	35.989
	2.0	98.600	98.318
0.2	0.2	20.359	21.176
	0.5	22.084	22.997
	1.0	32.255	33.671

	2.0	88.336	91.987
0.5	0.2	19.100	19.879
	0.5	20.718	21.588
	1.0	30.261	31.609
	2.0	82.873	86.354
1.0	0.2	18.094	18.832
	0.5	19.626	20.451
	1.0	28.666	29.945
	2.0	78.505	81.805
2.0	0.2	17.287	18.002
	0.5	18.751	19.549
	1.0	27.387	28.624
	2.0	75.005	78.199

Table 4 shows the non-dimensional frequencies of CCCC FG rectangular plates with different power law exponents, *n* and aspect ratios. It is clear that frequency parameters are increasing with increase in aspect ratios for a given power-law index. It is also noticeable that the frequencies are decreasing with increase in power-law indices for a given aspect ratio.

Table 5: The frequency parameters of square CCCC FG Al/Al_2O_3 plates with different power-law indices (n) and $k_w = 100$

n	Sources	Frequency parameters
0	Present study	37.701
	Chakraverty and Pradhan [5]	37.350
0.2	Present study	35.636
	Chakraverty and Pradhan [5]	35.033
0.5	Present study	33.719
	Chakraverty and Pradhan [5]	32.980
2.0	Present study	31.052
	Chakraverty and Pradhan [5]	30.066

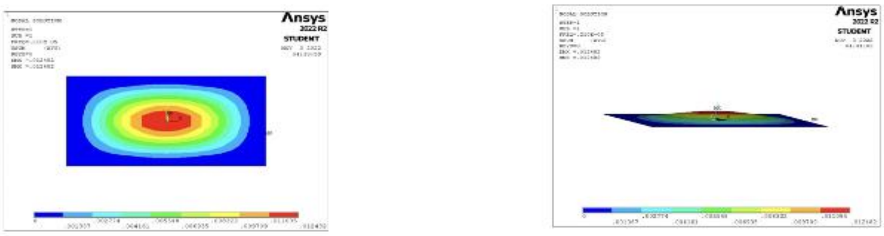
The effect of power law index on the frequency of vibration of CCCC FG plate resting on Winkler elastic foundation is very interesting. As shown in Table 5, the increase in power law index decreases the frequency parameters of the plate.

Table 6: Frequency parameters of CCCC FG Al/Al_2O_3 plates with different aspect ratios $(\frac{b}{a})$ ($n = 1$, $k_w = 100$)

$\frac{b}{a}$	Sources	Frequency parameters
0.2	Present study	21.649
	Chakraverty and Pradhan [5]	20.763
0.5	Present study	23.181
	Chakraverty and Pradhan [5]	22.266
1.0	Present study	32.221
	Chakraverty and Pradhan [5]	31.341
2.0	Present study	82.060
	Chakraverty and Pradhan [5]	82.9221

In Table 6, the comparison of the frequency parameters of CCCC FG plate with those reported by Chakraverty and Pradhan [5] using Rayleigh-Ritz method is presented for different aspect ratios. It is observed that with increase in aspect ratios, the frequency parameters increase.

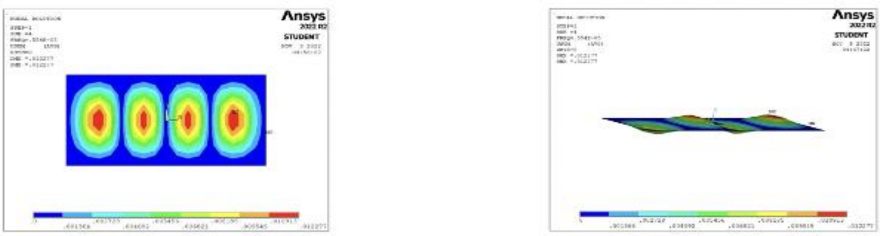
In Figures 4 and 5, the mode shapes of CCCC plate for aspect ratios 0.5 and 1, respectively, are shown.



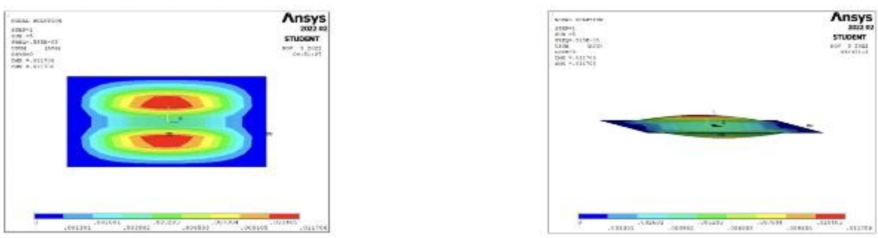
1st vibration mode

2nd vibration mode

3rd vibration mode

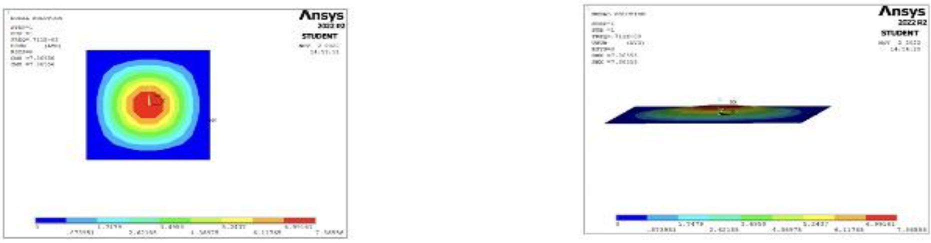


4th vibration mode

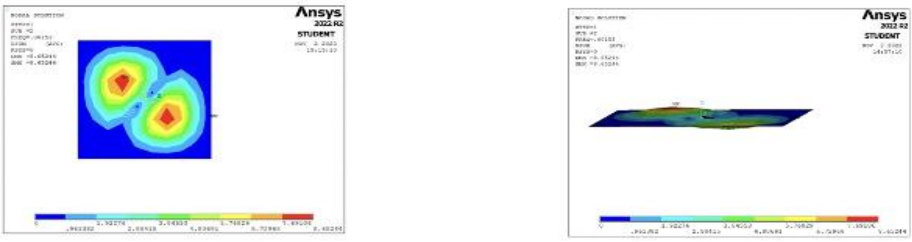


5th vibration mode

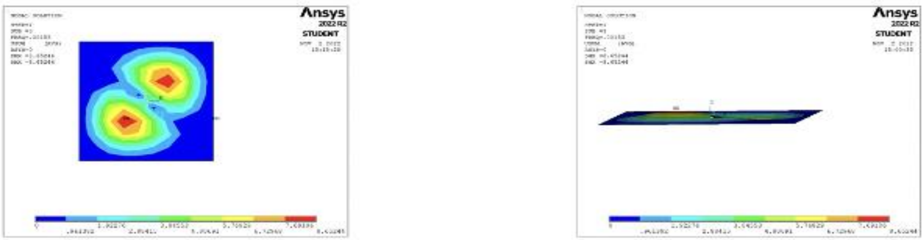
Figure 4: First five mode shapes of CCCC plates ($\beta = 0.5$)



1st vibration mode

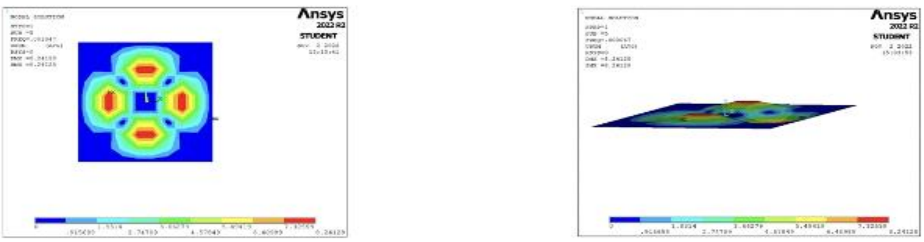


2nd vibration mode



3rd vibration mode

4th vibration mode



5th vibration mode

Figure 5: First five mode shapes of CCCC plates ($\beta = 1$)

IV. CONCLUSION

Integral calculus has been used in this work to evaluate the non-dimensional frequency parameters of isotropic FG rectangular plates sitting on Winkler elastic basis using the beam analogy method. The basic assumptions of Kirchhoff's linear, elastic, small-deflection theory of bending for thin plates are taken into account. It is evident that adding an elastic foundation increases the non-dimensional frequency parameter of the plates. Furthermore, like plates resting on Winkler foundation, an increase in aspect ratio, causes a corresponding increase in frequency for plates not subjected to the effect of Winkler elastic foundation regardless of the boundary configuration. So, the Winkler foundation parameter has a dominant influence on the frequencies of plates. The non-dimensional frequency parameter increases as the number of clamped edges in the plate increases. Hence, fixity of supports increases the fundamental natural frequency of the plate, and so increases the resistance of the plate to higher forcing frequencies before resonance can occur. It is also observed that an increase in power law index decreases the frequency parameters of the plate. These results clearly demonstrate the crucial role of stiffness defined by aspect ratios and boundary restraints. The model output was in close agreement with those of other researchers to about 99 percent accuracy.

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